A MULTISCALE METHOD FOR TURBULENT FLOW BASED ON LOCAL PROJECTION STABILIZATION

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Abstract. We propose to apply the recently introduced local projection stabilization for the Navier-Stokes equation to the numerical computation of turbulent flow. The discretization is done by nested finite element spaces. We show how this method may be cast in the framework of variational multiscale methods, indicating what the modelling assumptions are. Using a priori error estimation techniques inspired by the theory of stabilized finite element methods we prove the convergence of the method in the case of a linearized model problem. The a priori estimates are independent of the local Peclet number and give a sufficient condition for the size of the turbulent viscosity parameter in order to insure optimality of the approximation when the exact solution is smooth.

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1. Introduction. Today one of the major challenges in computational fluid dynamics is the accurate computation of different quantities in turbulent flow. The questions are many and the answers few. What can be computed? To what cost and to what precision? Therefore, it is not surprising that this subject is receiving increasing attention. Recently, several new approaches have been proposed such as the dynamic multilevel methology (DML) of Dubois, Jauberteau & Temam [9] or the variational multiscale method (VMS) of Hughes, Mazzei & Jansen, [13]. In the latter work, reference is made to residual free bubble techniques, see Brezzi & Russo [3], and subgrid viscosity as introduced by Guermond [11] to motivate an approach to Large-Eddy simulation (LES) where the turbulence model acts only on the fine scales.

Another very recent claim concerning the computation of turbulent flow is that certain average quantities such as mean drag can be computed on a PC using standard stabilized finite element methods of SUPG type together with adaptive mesh refinement controlled by a posteriori error estimation, see Johnson & Hoffmann [12]. This suggests a combined Large-Eddy / Direct-Numerical simulation (LES/DNS) approach for the computation of averaged quantities in turbulent flow. The situation remains somewhat contradictory since it was claimed in [13] that numerical evidence showed that SUPG alone fails to capture the characteristics of turbulent flow.

In any case, to be able to make a computation at all Reynolds numbers clearly one has to resort to a method that remains stable independent of the local Peclet number, unless of course a full DNS is aimed where the mesh is uniformly sufficiently fine to resolve all scales of the flow. So when working with finite elements the use of stabilized methods in one form or another seems mandatory for large eddy simulations. It is well known that the energy conservation property of the standard Galerkin formulation causes buildup of energy on the small scales which may lead to numerical instabilities.

In this paper, we advocate for the use of the two-level stabilization scheme for the Navier-Stokes equations (see Becker & Braack [2]). This is one in a group of more recently developed stabilized methods, as for instance Guermond [11], Becker & Braack [1], Burman & Hansbo [4, 5]. An attractive feature of this method for the computation of turbulent flow is that it can be cast in the framework of the variational

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multiscale method of [13] as we shall show. The stabilization is only acting on the smallest resolved scales of the flow. Other main advantages of this approach is that it shares similar conservation properties as a standard Galerkin finite element method. Moreover one does not need to resort to space-time finite elements for time stepping in order to stay consistent, but can apply any higher-order finite difference scheme for the discretization in time. By showing how this stabilized method can be formulated in the variational multiscale framework we give some additional evidence of the close relationship between stabilized methods and variational multiscale methods. We also prove optimal order a priori error estimates for the method, giving sufficient conditions on the length scales used in the subgrid viscosity model for the Galerkin projection method to remain stable. Finally, we will discuss the relation of this method to other stabilized finite element methods, and give some indications of its relation to GLS and residual-free bubbles.

In a forthcoming work we will give numerical evidence of the performance of the numerical scheme for some high Reynolds number flows in three space dimensions.

2. Variational formulation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a polygonal domain with boundary $\partial\Omega$. For the theoretical discussion, we consider the Navier-Stokes equations with, for simplicity, homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) - \mu\Delta v + \nabla p = f \quad & \text{in } \Omega\,, \\ \operatorname{div} v = 0 \quad & \text{in } \Omega\,, \\ v = 0 \quad & \text{on } \partial\Omega\,, \end{aligned}$$

subject to some initial condition $v(\cdot,0) = v_0$. Above $v = v(x,t)$ denotes the velocities and $p = p(x,t)$ the pressure. The gradient in space is denoted by ∇, the divergence with respect to space by div . Let $I := [0,T]$ be the time interval, V the Sobolev space $V := [H_0^1(\Omega)]^d$, and Q the space of square-integrable functions with zero mean, $Q := L_0^2(\Omega)$. The product space is denoted by $X = V \times Q$. The velocities are sought in the Bochner space $\mathcal{V}^v := H^1(I,V)$, and the pressure in $\mathcal{V}^p := L^2(I,Q)$. The product space will be denoted by $\mathcal{V} := \mathcal{V}^v \times \mathcal{V}^p$. The test functions are in the space $\mathcal{W} := L^2(I,X)$. The L^2-scalar product over the space-time slab $\Omega_T := \Omega \times I$ will be denoted by $(\cdot,\cdot)$, and its norm by $\|\cdot\|$. Introducing now the state vector $u = \{v,p\} \in u_0 + \mathcal{V}$ with a prolongation u_0 of the initial data v_0 for the velocities, we may write the standard variational formulation: Find $u \in u_0 + \mathcal{V}$ such that

$$B(u,\varphi) = (f,\psi) \quad \forall\varphi = \{\psi,\xi\} \in \mathcal{W}\,, \tag{2.1}$$

where $B(u,\varphi)$ is defined by

$$B(u,\varphi) := (\partial_t v,\psi) - (v\otimes v,\nabla\psi) + (\mu\nabla v,\nabla\psi) - (p,\operatorname{div}\psi) + (\operatorname{div} v,\xi)\,.$$

3. Separation of scales on the continuous level. The local projection method that we are going to use can be cast in the framework of the variational multiscale method. In the VMS as introduced in [13], a scale separation is performed and the turbulence model acts only on the finer scales. However, as always in turbulence modeling certain model assumptions on the interaction between the scales are made.

To make clear what our model assumptions are, we use the three-level partition proposed in Collis [8]. Hence we consider a scale separation in large resolved scales, denoted by $\bar{u}$, small resolved scales denoted by $\tilde{u}$ and unresolved scales denoted by $\hat{u}$. The solution space is partitioned in a corresponding manner

$$\mathcal{V} = \bar{\mathcal{V}} \oplus \tilde{\mathcal{V}} \oplus \hat{\mathcal{V}}.$$

The function space $\mathcal{W}$ is partitioned similarly, $\mathcal{W} = \bar{\mathcal{W}} \oplus \tilde{\mathcal{W}} \oplus \hat{\mathcal{W}}$, with corresponding test functions, for instance, $\bar{\varphi} = (\bar{\psi}, \bar{\xi}) \in \bar{\mathcal{W}}$. We now write the exact equations of motions for each scale

$$B(u, \bar{\varphi}) = (f, \bar{\psi}) \quad \forall \bar{\varphi} \in \bar{\mathcal{W}}\,, \tag{3.1}$$
$$B(u, \tilde{\varphi}) = (f, \tilde{\psi}) \quad \forall \tilde{\varphi} \in \tilde{\mathcal{W}}\,, \tag{3.2}$$
$$B(u, \hat{\varphi}) = (f, \hat{\psi}) \quad \forall \hat{\varphi} \in \hat{\mathcal{W}}\,. \tag{3.3}$$

Introducing the linearized Navier-Stokes operator

$$\begin{aligned} B'(u, u', \varphi) := (\partial_t v', \hat{\psi}) - (v' \otimes v + v \otimes v', \nabla \psi) \\ -(p', \nabla \cdot \psi) + (\mu \nabla v', \nabla \psi) + (\nabla \cdot v', \xi), \end{aligned}$$

the Reynolds stress projection

$$R(v, \psi) := (v \otimes v, \nabla \psi)\,,$$

and the cross stress projection operator

$$C(v, \hat{v}, \psi) := (v \otimes \hat{v} + \hat{v} \otimes v, \nabla \psi)\,,$$

we may reformulate the exact equations for each scale in a fashion that makes evident the coupling between the scales. Following Collis [8], the exact solution for the resolved large scales fulfills for all $\bar{\varphi} \in \bar{\mathcal{W}}$ the equation

$$\begin{aligned} B(\bar{u}, \bar{\varphi}) + B'(\bar{u}, \tilde{u}, \bar{\varphi}) - R(\tilde{v}, \bar{\psi}) = (f, \bar{\psi}) \\ - B'(\bar{u}, \hat{u}, \bar{\varphi}) - R(\hat{v}, \bar{\psi}) + C(\tilde{v}, \hat{v}, \bar{\psi})\,. \end{aligned} \tag{3.4}$$

The first line in (3.4) includes the influence of the resolved scales on the large scales, whereas the second line includes the influence of the unresolved scales on the large scales. In the same fashion, the small resolved scales fulfill for all $\tilde{\varphi} \in \tilde{\mathcal{W}}$:

$$\begin{aligned} B'(\bar{u}, \tilde{u}, \tilde{\varphi}) - R(\tilde{v}, \tilde{\psi}) = (f, \bar{\psi}) - B(\bar{u}, \tilde{\varphi}) \\ - B'(\bar{u}, \hat{u}, \tilde{\varphi}) - R(\hat{v}, \tilde{\psi}) + C(\tilde{v}, \hat{v}, \tilde{\psi})\,. \end{aligned} \tag{3.5}$$

Note that the large scale residual is driving the fine scales and that the form of the influence of the unresolved scales on the resolved scales take the same form for large resolved scales as for small resolved scales. The unresolved scales finally satisfy the following equation for all $\hat{\varphi} \in \hat{\mathcal{W}}$

$$B'(\bar{u} + \tilde{u}, \hat{u}, \hat{\varphi}) + R(\hat{v}, \hat{\psi}) = (f, \hat{\psi}) - B(\bar{u} + \tilde{u}, \hat{\varphi})\,.$$

It follows that the equation for the unresolved scales is driven by the residual of the resolved scales. With the equations written in this form it is easy to state what the modeling assumptions are:

(M1) The unresolved scales $\hat{u}$ have no "direct" influence on the large scales. This means that the second line of equation (3.4) is set to zero:

$$-B'(\bar{u}, \hat{u}, \bar{\varphi}) - R(\hat{v}, \bar{\psi}) + C(\tilde{v}, \hat{v}, \bar{\psi}) = 0 \quad \forall \bar{\varphi} \in \bar{\mathcal{W}}. \tag{3.6}$$

(M2) The influence of the unresolved scales on the small scales is modeled by an artificial viscosity term

$$S : (\bar{\mathcal{V}} \oplus \tilde{\mathcal{V}}) \times (\bar{\mathcal{V}} \oplus \tilde{\mathcal{V}}) \to \mathbb{R},$$

acting only on the small resolved scales. Hence we assume in (3.5) that for $\tilde{\varphi} \in \tilde{\mathcal{W}}$:

$$S(\tilde{u}, \tilde{\varphi}) \approx B'(\bar{u}, \hat{u}, \tilde{\varphi}) + R(\hat{v}, \tilde{\psi}) - C(\tilde{v}, \hat{v}, \tilde{\psi})\,. \tag{3.7}$$

The first modeling assumption (M1) can be expected to hold true when the main features of the flow are resolved. This is the large eddy assumption. The second modeling assumption (M2) seems to imply that the space of the resolved small scales should be sufficiently big with respect to the space of the large scales. Otherwise why should the behavior be any different? This is not so promising from a computational viewpoint. Since the large scales have to resolve the main features of the flow (that we want to compute) we want to take the space of small resolved scales as small as possible as it only represents the fluctuations. However one may argue that if assumption (M1) is satisfied then the exact form or size of the subgrid model is of less importance *as long as it allows for a sufficient rate of dissipation of energy from the resolved small scales to the unresolved scales.* Insufficient dissipation will cause buildup of energy on the resolved small scales (by the conservation properties of the Galerkin method) leading to spurious oscillations and eventually divergence. Excessive dissipation will cause too much damping of the resolved small scales which will lead to poorer resolution of the large scales through the Reynolds stress coupling.

Using these modeling assumptions we arrive at the formulation

$$\begin{array}{lcll} B(\bar{u} + \tilde{u}, \bar{\varphi}) & = & (f, \bar{\psi}) & \forall \bar{\varphi} \in \bar{\mathcal{W}}\,, \\ B(\bar{u} + \tilde{u}, \tilde{\varphi}) + S(\tilde{u}, \tilde{\varphi}) & = & (f, \tilde{\psi}) & \forall \tilde{\varphi} \in \tilde{\mathcal{W}}. \end{array} \tag{3.8}$$

We choose the subgrid viscosity term in such a way that

1) it is coercive on the small resolved scales $\tilde{u}$

$$S(\tilde{u}, \tilde{u}) \geq c \|\nabla \tilde{u}\|^2 \quad \forall \tilde{u} \in \tilde{\mathcal{W}}\,,$$

2) it is symmetric

$$S(u, \varphi) = S(\varphi, u) \quad \forall u, \varphi \in \bar{\mathcal{W}} \oplus \tilde{\mathcal{W}}\,, \tag{3.9}$$

3) it vanishes on the large resolved scales

$$S(\bar{\varphi}, \cdot) = 0, \quad \forall \bar{\varphi} \in \bar{\mathcal{W}}. \tag{3.10}$$

The resolved scales will be represented by nested finite element spaces. To this end we introduce some finite element approximation $\mathcal{V}_h$ of $\mathcal{V}$ that will represent the resolved scales $\mathcal{V}_h = \bar{\mathcal{V}} \oplus \tilde{\mathcal{V}}$. We now introduce the corresponding decomposition of the discrete space in large and small resolved scales. To emphasize the fact that the resolved scales are discrete spaces we may supplement their notation with an h for the small resolved scales and an H for the large resolved scales so that $\mathcal{V}_H \equiv \bar{\mathcal{V}}$. Since the subgrid model will depend on the mesh size, it will be denoted in the following with an subscript, $S_h(\cdot, \cdot)$. The same approximation is done with the test space $\mathcal{W}_h = \bar{\mathcal{W}} \oplus \tilde{\mathcal{W}}$. Backtracking the whole argument to the original weak formulation (2.1) and using the modeling assumptions (M1) and (M2) we arrive at the formulation, find $u_h \in u_0 + \mathcal{V}_h$ such that

$$B(u_h, \varphi) + S_h(\tilde{u}_h, \tilde{\varphi}) = (f, \psi) \quad \forall \varphi \in \mathcal{W}_h \,, \tag{3.11}$$

or using the scale separation property (3.10) of $S_h(\cdot,\cdot)$

$$B(u_h, \varphi) + S_h(u_h, \varphi) = (f, \psi) \quad \forall \varphi \in \mathcal{W}_h \,. \tag{3.12}$$

Note also that by the properties of $S_h(\cdot,\cdot)$ we have Galerkin orthogonality for the discretization error $u - u_h$ on the large resolved scales:

$$B(u - u_h, \bar{\varphi}) = 0 \quad \forall \bar{\varphi} \in \mathcal{W}_H \,. \tag{3.13}$$

So far we have kept the discussion on an abstract level, we will end this section with some remarks discussing our specific choice of subgrid model. As proposed by Hughes et. al. [13] the subgrid model $S_h(\tilde{u}, \tilde{\varphi})$ will be chosen as an artificial viscosity type operator acting only on the small scale. Instead of using a Smagorinsky type model we will use the simple linear model proposed in the two level formulation of [2]. For the choice of the artificial viscosity parameter in $S_h(\cdot,\cdot)$ we will rely on a priori error estimates that are uniform in the Reynolds number and valid for smooth solutions u rather than parameters obtained from experimental data or heuristics.

Furthermore, a new feature of this model is that the pressure fluctuations are included in the subgrid model. This can be motivated by the interaction of unresolved pressure fluctuation and small scale velocity in the right-hand side of (3.7). Moreover, as shown in [1] it allows us to use equal-order interpolation for the approximation of velocities and pressure. This is convenient since there is no reason to take the resolved small scales of the velocity differently than the resolved small scales of the pressure.

4. Discretization by finite elements. In fact the basic ideas behind the discretization in space has already been outlined in the previous section. What remains is the exact choice of finite element spaces, of the subgrid model, and of the discretization scheme in time. We will give special focus on the case of d-linear elements but the analysis extends to d-quadratics in the obvious way.

In the following we consider shape regular meshes $\mathcal{T}_h = \{K\}$ of hexaedral elements K with the minimum mesh size $h = \min\{h_K : K \in \mathcal{T}_h\}$ (quadrilateral elements for the academical case $d = 2$). We use the finite element spaces P_h^r resulting from $r-$linear transformations of $r-$linear polynomials $\hat{\varphi}$ on a reference cell $\hat{K}$:

$$P_h^r(\Omega, \mathbb{R}) := \{\varphi \in C(\Omega, \mathbb{R}) : \varphi|_K = \hat{\varphi} \circ T_K^{-1}\}\,.$$

The discontinuous analogon is denoted by $P_{h,disc}^r$,

$$P_{h,disc}^r(\Omega, \mathbb{R}) := \{\varphi \in L^2(\Omega, \mathbb{R}) : \varphi|_K = \hat{\varphi} \circ T_K^{-1}\}\,.$$

We will treat d-linear elements ($r = 1$) and d-quadratic ($r = 2$) elements simultaneously in the analysis. These finite element spaces will be called simply Q_1 in the case of $r = 1$, and Q_2 elements in the case $r = 2$. The discrete pressure space Q_h is the subspace of P_h^r with zero mean, and the velocity space V_h is the subspace with vanishing trace:

$$Q_h := \left\{ \xi \in P_h(\Omega, \mathbb{R}); \int_\Omega \xi \, dx = 0 \right\} , \quad V_h := \{\psi \in P_h^r(\Omega, \mathbb{R}^d); \psi|_{\partial\Omega} = 0\}\,.$$

The product space is denoted by X_h:

$$X_h := V_h \times Q_h \,.$$

Furthermore, for $q \in \mathbb{N}$, $H := 2^q h$ let $\mathcal{T}_H$ be the coarser mesh obtained by q times "global coarsening" of $\mathcal{T}_h$. Obviously, the finer mesh $\mathcal{T}_h$ contains 2^{dq} times more elements than $\mathcal{T}_H$. The corresponding finite element spaces are denoted by $Q_H \subset Q_h$ and $V_H \subset V_h$.

Let us partition the time interval I into subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, \dots, N$ with $0 = t_0 < t_1 < \dots < t_N = T$ and $\tau_n := t_n - t_{n-1}$. We also introduce the space time slabs $Q_n := I_n \times \Omega$. As time integration scheme, we use the Crank-Nicholson scheme. It means that we choose piecewise linears for the ansatz functions, precisely:

$$\bar{\mathcal{V}}_h^v := P_\tau^1(I, V_H)\,, \quad \tilde{\mathcal{V}}^v := P_\tau^1(I, V_h \setminus V_H)\,, \quad \hat{\mathcal{V}}^v := \mathcal{V}^v \setminus P_\tau^1(I, V_h)\,,$$

and as test spaces piecewise constants (discontinuous):

$$\bar{\mathcal{W}}_h^v := P_\tau^0(I, V_H)\,, \quad \tilde{\mathcal{W}}^v := P_\tau^0(I, V_h \setminus V_H)\,, \quad \hat{\mathcal{W}}^v := \mathcal{W}^v \setminus P_\tau^0(I, V_h)\,.$$

The pressure spaces $\bar{\mathcal{V}}_h^p, \tilde{\mathcal{V}}^p$ and $\hat{\mathcal{V}}^p$ are defined analogously by the use of Q_H and Q_h. The total ansatz and test spaces of resolved scales are denoted by

$$\begin{aligned}
\mathcal{V}_h &= (\bar{\mathcal{V}}^v \oplus \tilde{\mathcal{V}}^v) \times (\bar{\mathcal{V}}^p \oplus \tilde{\mathcal{V}}^p) = P_\tau^1(I, X_h)\,,\\
\mathcal{W}_h &= (\bar{\mathcal{W}}^v \oplus \tilde{\mathcal{W}}^v) \times (\bar{\mathcal{W}}^p \oplus \tilde{\mathcal{W}}^p) = P_\tau^0(I, X_h)\,.
\end{aligned}$$

With these finite element spaces we now propose the following finite element method: Find $u_h \in u_0 + \mathcal{V}_h$, so that in the n-th time step it holds for the restriction $u^n = \{v^n, p^n\} := u_h|_{I_n}$:

$$(\tau_n^{-1} v^n, \psi) + A(u^n, \varphi) + S_h(u^n, \varphi) = g_n(u^{n-1}, \varphi) \quad \forall \varphi \in \mathcal{W}_h\,, \tag{4.1}$$

with

$$\begin{aligned}
A(u^n, \varphi) &:= -(v \otimes v, \nabla\psi) + (\mu \nabla v, \nabla \psi) - (p, \operatorname{div} \psi) + (\operatorname{div} v, \xi)\\
g_n(u^{n-1}, \varphi) &:= (f, \psi) + (\tau_n^{-1} v^{n-1}, \psi) - (\mu \nabla v^{n-1}, \nabla \psi)\\
&\quad + (v^{n-1} \otimes v^{n-1}, \nabla \psi) - S_h(u^{n-1}, \varphi)\,.
\end{aligned}$$

In order to specify the subgrid model $S_h(\cdot,\cdot)$, we have to introduce further notations. Let D_h^v and D_h^p be the following space for pressure and velocities, respectively, of functions allowing discontinuities across elements of $\mathcal{T}_{2h}$:

$$\begin{aligned}
D_h^v &:= [P_{2h,disc}^{r-1}(\Omega, \mathbb{R}^d)]^d\,,\\
D_h^p &:= P_{2h,disc}^{r-1}(\Omega, \mathbb{R})\,.
\end{aligned}$$

In the case $r = 1$, these spaces contain patch-wise ($K \in V_{2h}$) constants; and for $r = 2$, they contain patch-wise d-linear elements. We will make use of the L^2-projection operator (in space)

$$\bar{\pi}_h : P_\tau^s(I, L^2(\Omega)) \to P_\tau^s(I, D_h^p)\,, \quad s \in \{0, 1\}\,.$$

The operator giving the space fluctuations is denoted by

$$\bar{\varkappa}_h := i - \bar{\pi}_h\,,$$

with the identity mapping i. Similarly, we define the (space) nodal interpolant $\pi_h : P_\tau^s(I, Q_h) \to P_\tau^s(I, Q_{2h})$ and $\varkappa_h := i - \pi_h$. We use the same notations $\bar\pi_h$, $\bar\varkappa_h, \pi_h$ and $\varkappa_h$ for the mappings on vector-valued functions, for instance, $\bar\pi_h : P_\tau^s(I, L^2(\Omega)^d) \to P_\tau^s(I, D_h^v)$.

With these notations, we may take as subgrid model

$$S_h^{\varkappa}(u,\varphi) := (\mu_T \nabla \varkappa_h v, \nabla \varkappa_h \psi) + (\nu_T \nabla \varkappa_h p, \nabla \varkappa_h \xi)\,, \tag{4.2}$$

or

$$S_h^{\bar\varkappa}(u,\varphi) := (\mu_T \bar\varkappa_h \nabla v, \bar\varkappa_h \nabla \psi) + (\nu_T \bar\varkappa_h \nabla p, \bar\varkappa_h \nabla \xi)\,, \tag{4.3}$$

with parameters μ_T and ν_T depending on h. Clearly on tetrahedral nested finite elements (so-called P_1 or P_2 elements) both subgrid operators $S_h^{\varkappa}(\cdot,\cdot)$ and $S_h^{\bar\varkappa}$ satisfies (3.10) exactly. In fact in this case one easily shows that they are equivalent. However, for the Q_1 or Q_2 elements considered here, using the choice (4.3) a small residual may remain due to the cross-term. Consequently, we do not have exact scale separation for (4.3) on hexaedrals (quadrilaterals in two space dimensions). On the other hand, the second form has advantages from a theoretical viewpoint. We will exploit this in the next section to choose the subgrid viscosity parameter in such a way that the convergence of the method is optimal whenever the underlying solution is smooth. It should also be noted that from the practical viewpoint it may be more advantageous to use the streamline derivative in the part of the subgrid model acting on the velocity in order to minimize cross-wind diffusion, for instance,

$$\begin{aligned} S_h^{\beta}(u,\varphi) := (\mu_T \bar\varkappa_h (\beta\cdot\nabla) v, \bar\varkappa_h (\beta\cdot\nabla)\psi) + (\mu_T \bar\varkappa_h \operatorname{div} v, \bar\varkappa_h \operatorname{div}\psi) \\ + (\nu_T \bar\varkappa_h \nabla p, \bar\varkappa_h \nabla \xi)\,. \end{aligned} \tag{4.4}$$

However, as we shall see in the next section, this does not affect the order of the numerical scheme.

5. A priori error analysis. To tune the parameter of the subgrid model we use a priori error estimation on a linear stationary model problem (of Oseen type) instead of modeling assumptions. Assuming sufficient regularity of the underlying solution the parameters are chosen in such a way that the method has optimal convergence properties independently of the viscosity. The analysis also serves as a priori analysis for the numerical scheme (4.1). We only treat the interesting case of high Reynolds number, hence assuming that $\mu \le |\beta| h$. First we prove an estimate for a mesh-depending norm $\|\!|\cdot|\!\|$ including the H^1-norm of the velocities and the subgrid model error. We then use this estimate to recover control of the pressure and show that the L^2-norm error of the pressure is bounded by the triple norm of the error of the state vector u_h. We refer to [2, 6] for more details on a priori error estimate for similar discretizations applied to the (linearized) Navier-Stokes equation. The simplified model we propose takes the form

$$\begin{aligned} \sigma v + \operatorname{div}(\beta \otimes v) - \mu \Delta v + \nabla p &= f \quad \text{in } \Omega, \\ \nabla \cdot v &= 0 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{5.1}$$

with some given solenoidal vector field β and $\sigma > 0$. This problem is well-posed in the space $V \cap H_0(\operatorname{div};\Omega)$ by the Lax-Milgram Lemma. The L^2-scalar product over

Ω will be simply denoted by $(\cdot,\cdot)_\Omega$ and the corresponding norm by $\|\cdot\|_\Omega$. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_{s,\Omega}$.

We consider the following two-level finite element formulation, find $u_h \in X_h$ such that

$$A(u_h,\varphi) + S_h(u_h,\varphi) = (f,\psi)_\Omega \quad \forall \varphi \in X_h\,, \tag{5.2}$$

where now

$$A(u,\varphi) := (\sigma v,\psi)_\Omega - (\beta\otimes v,\nabla\psi)_\Omega - (p,\operatorname{div}\psi)_\Omega + (\operatorname{div} v,\xi)_\Omega + (\mu\nabla v,\nabla\psi)_\Omega\,.$$

The subgrid model is given as in (4.3):

$$S_h(u,\varphi) := (\mu_T \bar{\varkappa}_h \nabla v, \bar{\varkappa}_h\nabla\psi)_\Omega + (\nu_T \bar{\varkappa}_h\nabla p, \bar{\varkappa}_h\nabla\xi)_\Omega\,. \tag{5.3}$$

with the fluctuation filter $\bar{\varkappa}_h = i - \bar{\pi}_h$ and $\bar{\pi}_h : L^2(\Omega) \to D_h^p$ or $\bar{\pi}_h : L^2(\Omega)^d \to D_h^v$, defined analogously to the previous section. In the following analysis, we make use of the interpolation and stability property of $\bar{\varkappa}_h$.

LEMMA 5.1.

$$\|\bar{\varkappa}_h\nabla v\|_\Omega \lesssim h\|v\|_{2,\Omega} \quad \forall v\in H^2(\Omega)\,, \tag{5.4}$$

$$\|\bar{\varkappa}_h v\|_\Omega \lesssim \|v\|_\Omega \quad \forall v \in L^2(\Omega)\,. \tag{5.5}$$

Proof. The interpolation property (5.4) is an immediate consequence of the patch-wise interpolation of $\bar{\pi}_h$ for the H^1 function $w := \nabla v$:

$$\|\bar{\varkappa}_h\nabla v\|_K = \|w - \bar{\pi}_h w\|_K \lesssim h_K\|w\|_{1,K} \le h_K\|v\|_{2,K} \quad \forall K\in\mathcal{T}_{2h}\,.$$

Stability of $\bar{\varkappa}_h$ is due to the L^2-stability of $\bar{\pi}_h$:

$$\|\bar{\varkappa}_h v\|_\Omega \le \|v\|_\Omega + \|\bar{\pi}_h v\|_\Omega \lesssim \|v\|_\Omega\,.$$

□

We will prove under the assumption of sufficiently regular pressure and velocity $v\in H_0^2(\Omega)^3$, $p\in H^2(\Omega)\cap L_0^2(\Omega)$, that a certain scaling of μ_T and ν_T gives optimal convergence of the velocities independent of the Reynolds number. A similar result is then proven for the L^2-norm of the pressure.

To this end we introduce the triple norm in the space X:

$$|\!|\!| u |\!|\!| = |\!|\!| \{v,p\} |\!|\!| := \left(\|\sigma^{1/2}v\|_\Omega^2 + \|\mu^{1/2}\nabla v\|_\Omega^2 + S(u,u)\right)^{1/2}\,.$$

By the following coercivity result we deduce existence and uniqueness of the discrete velocities.

LEMMA 5.2. *We have the following coercivity property:*

$$|\!|\!| u |\!|\!|^2 = A(u,u) + S_h(u,u) \quad \forall u\in X\,. \tag{5.6}$$

Proof. The proof follows immediately by integration by parts. □

We have the following approximate Galerkin orthogonality

LEMMA 5.3. *Let $u \in X$ be the solution of the weak formulation of (5.1) and $u_h \in X_h$ the solution of its discrete version (5.2). Then it holds:*

$$A(u - u_h, \varphi) = -S_h(u_h, \varphi) \quad \varphi \in X_h\,.$$

Proof. The proof is obtained by subtracting (5.2) from the weak formulation of (5.1). ☐

Since the method is not strongly consistent in the sense that we do not have full Galerkin orthogonality, we must analyze the asymptotic behavior of the subgrid model, i.e., the dependence with respect to the mesh size h. We will use the notation $\lesssim$ to indicate that there may arise (mesh independent) constants in the estimates. We state a result for a modified Clément interpolation operator introduced in [1] with a generalization to Q_2 elements.

LEMMA 5.4. *There is an interpolation operator*

$$j_h : V \to V_h$$

with the orthogonality property

$$(v - j_h v, \psi)_\Omega = 0, \quad \forall \psi \in D_h^v\,, \forall v \in V\,, \tag{5.7}$$

having optimal approximation properties in the L^2-norm and H^1-seminorm

$$\|v - j_h v\|_\Omega \lesssim h^r \|v\|_{r,\Omega} \quad \forall v \in [H^r(\Omega)]^d\,, \tag{5.8}$$

$$\|\nabla(v - j_h v)\|_\Omega \lesssim h^{r-1} \|v\|_{r,\Omega} \quad \forall v \in [H^r(\Omega)]^d\,, \tag{5.9}$$

with $r \in \{1, 2\}$, and is L^2- and H^1-stable:

$$\|j_h v\|_{s,\Omega} \lesssim \|v\|_{s,\Omega} \quad \forall v \in [H^s(\Omega)]^d,\ s \in \{0, 1\}\,. \tag{5.10}$$

Proof. The construction uses the Clément interpolation operator $j_h^{Cl} : V \to V_h$, see Clément [7], which already fulfills the approximation properties (5.8), (5.9), maintains homogeneous Dirichlet values, and has the stability property (5.10). In order to ensure (5.7), we modify j_h^{Cl} by a projection $m_h : V \to V_h$,

$$j_h = j_h^{Cl} + m_h\,.$$

For the definition of m_h, we choose a suitable basis of D_h^v, consisting of r^2 test functions $\psi_{K,i}$, $i = 1, \dots, r^d$ for each patch $K \in \mathcal{T}_{2h}$, and certain basis functions $\phi_{K,i} \in V_h$, $i = 1, \dots, r^d$ with support in K. The specific form of these test functions depends of cause on the polynomial degree r. However, the perturbation $m_h v$ will be of the form

$$m_h v = \sum_{K \in \mathcal{T}_{2h}} \sum_{i=1}^{r^d} \alpha_{K,i}(v) \phi_{K,i}\,,$$

with appropriate coefficients $\alpha_{K,i}(v)$, so that it fulfills (5.7). In other words:

$$\int_K m_h v\, \psi_{K,i}\, dx = \int_K (v - j_h^{Cl} v) \psi_{K,i}\, dx \quad \forall K \in \mathcal{T}_{2h}\, \forall i = 1, \dots, r^d\,. \tag{5.11}$$

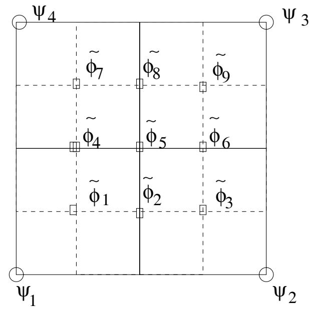

FIG. 5.1. *A patch K of four Q_2 cells in two dimensions with the bilinears ψ_i and the quadratic bubbles $\tilde\phi_i$, $i = 1, \dots, 9$ used in the proof of Lemma 5.4*

This property is equivalent to solving for each $K \in \mathcal{T}_{2h}$ a linear system with the matrix $M = (m_{ij})$, $i, j = 1, \dots, r^d$ and entries

$$m_{ij} := \int_K \psi_{K,i}\phi_{K,j}\, dx\,.$$

Let $\tilde\phi_{K,l}$, $l = 1, \dots, (2r-1)^d$ denote the nodal basis functions corresponding to the interior degrees of freedom of K, see Figure 5.1. We now choose the basis $\phi_{K,i}$ as being the L^2-projection of $\psi_{K,i}$ onto these basis functions. Find $\phi_{K,i} = \sum_l \tilde\alpha_{i,l}\tilde\phi_{K,l}$ such that

$$(\phi_{K,i}, \tilde\phi_{K,l})_K = (\psi_{K,i}, \tilde\phi_{K,l})_K, \quad l = 1, \dots, (2r-1)^d.$$

With this choice of basis we note that there holds

$$\begin{aligned}(\psi_{K,i}, \phi_{K,j},)_\Omega &= \sum_l \tilde\alpha_{j,l}(\psi_{K,i}, \tilde\phi_{K,l})_\Omega \\ &= \sum_l \tilde\alpha_{j,l}(\phi_{K,i}, \tilde\phi_{K,l})_K \\ &= (\phi_{K,i}, \phi_{K,j})_\Omega\,.\end{aligned}$$

It follows that M is symmetric positive definite and that on each patch $K \in \mathcal{T}_{2h}$ there holds:

$$\|m_h v\|_K^2 \lesssim \int_K m_h v\, \bar\pi_h(m_h v)\, dx\,. \tag{5.12}$$

For proving the approximation properties (5.8) and (5.9) we estimate on each patch $K \in \mathcal{T}_{2h}$ due to (5.12):

$$\begin{aligned}\|m_h u\|_K^2 &\lesssim \int_K (v - j_h^{Cl} v)\bar\pi_h m_h v\, dx \\ &\le \|v - j_h^{Cl} v\|_K\, \|\bar\pi_h m_h v\|_K \\ &\lesssim \|v - j_h^{Cl} v\|_K\, \|m_h v\|_K\,,\end{aligned}$$

where we used the L^2 stability of $\bar\pi_h$ in the last inequality. It follows $\|m_h v\|_K \lesssim \|v - j_h^{Cl} v\|_K$. Consider now the approximation in the L^2-norm, clearly

$$\begin{aligned}\|v - j_h v\|_\Omega &= \|v - j_h^{Cl} v + m_h v\|_\Omega \\ &\le \|v - j_h^{Cl} v\|_\Omega + \|m_h v\|_\Omega \\ &\lesssim \|v - j_h^{Cl} v\|_\Omega\,.\end{aligned}$$

For proving (5.9) we proceed in a similar fashion by applying an inverse estimate:

$$\begin{aligned}\|\nabla m_h v\|_\Omega^2 &\lesssim \sum_{K\in\mathcal{T}_h} h_K^{-2}\|m_h v\|_K^2 \\ &= \sum_{K\in\mathcal{T}_h} h_K^{-2}\|v - j_h^{Cl} v\|_K^2 \\ &\lesssim \sum_{K\in\mathcal{T}_h} h_K^{2(r-1)}\|v\|_{r,\widetilde{K}}^2 \\ &\lesssim h^{2(r-1)}\|v\|_{r,\Omega}^2 \,.\end{aligned}$$

□

Remark: The interpolation operator j_h^{Cl} maintains *homogeneous* Dirichlet conditions on (parts of) $\partial\Omega$. For polynomial Dirichlet condition or higher order approximations, the interpolation introduced by Melenk & Wohlmuth [14] can be used.

The interpolation operator j_h acts on the velocity space, but the result holds of cause true for the scalar space $L^2(\Omega)$. We will use, therefore, the notation j_h also for the interpolation operator acting on the state variable $u = \{v, p\}$.

LEMMA 5.5. *For the interpolation operator j_h of the previous Lemma we have for all $u \in X \cap [H^r(\Omega)]^{d+1}$ with $r \in \{1, 2\}$:*

$$S_h(j_h u, j_h u)^{1/2} \lesssim (\mu_T^{1/2} + \nu_T^{1/2}) h^{r-1}(\|v\|_{r,\Omega} + \|p\|_{r,\Omega}) \,.$$

Proof. We start with adding and subtracting u:

$$\begin{aligned}S_h(j_h u, j_h u) &= S_h(u + j_h u - u, u + j_h u - u) \\ &\le S_h(u,u) + S_h(j_h u - u, j_h u - u) + 2S_h(j_h u - u, u) \\ &\le 2(S_h(u,u) + S_h(j_h u - u, j_h u - u)).\end{aligned}$$

For the first term the result follows immediately by the interpolation property (5.4):

$$\begin{aligned}S_h(u,u) &\le \mu_T\|\bar{\varkappa}_h \nabla v\|_\Omega^2 + \nu_T\|\bar{\varkappa}_h \nabla p\|_\Omega^2 \\ &\lesssim \mu_T h^{2(r-1)}\|v\|_{r,\Omega}^2 + \nu_T h^{2(r-1)}\|p\|_{r,\Omega}^2 \,.\end{aligned}$$

For the second term $S(j_h u - u, j_h u - u)$ we have:

$$\begin{aligned}\mu_T\|\bar{\varkappa}_h \nabla(j_h v - v)\|_\Omega^2 &\lesssim \mu_T\|\nabla(j_h v - v)\|_\Omega^2 \\ &\lesssim \mu_T h^{2(r-1)}\|v\|_{r,\Omega}^2 \,,\end{aligned}$$

using the L^2-stability (5.5) of the local projector $\bar{\varkappa}_h$ and the interpolation property (5.9) of j_h. For the pressure contribution of course the same holds. □

The following Lemma states the fact that the proposed stabilization term (4.4) involving only diffusion in in streamline direction can be bounded by the triple norm.

LEMMA 5.6. *If $\beta \in [W^{1,\infty}(\Omega)]^d$, then it holds for all $v \in V_h$*

$$\|\mu_T^{1/2}\bar{\varkappa}_h(\beta\cdot\nabla)v\|_\Omega \le C_\beta \|\{v, 0\}\| \,, \tag{5.13}$$

where $C_\beta \sim \mu_T^{1/2}\|\beta\|_{W^{1,\infty}(\Omega)}\sigma^{-1/2} + \|\beta\|_{\infty,\Omega}$.

Proof. The proof follows by adding and subtracting $\bar{\pi}_h\beta$, where $\bar{\pi}_h$ denotes the projection on D_h^v (here denoting the space of piecewise constants on the macro patches, regardless of the approximation). We apply the triangle inequality and the H^1 stability of $\bar{\varkappa}_h$

$$\begin{aligned}\|\mu_T^{1/2}\bar{\varkappa}_h(\beta\cdot\nabla)v\|_\Omega &\leq \|\mu_T^{1/2}\bar{\varkappa}_h((\beta-\bar{\pi}_h\beta)\cdot\nabla)v\|_\Omega+\|\mu_T^{1/2}\bar{\varkappa}_h((\bar{\pi}_h\beta)\nabla)v\|_\Omega\\ &\lesssim \|\mu_T^{1/2}((\beta-\bar{\pi}_h\beta)\cdot\nabla)v\|_\Omega+\|\mu_T^{1/2}\bar{\varkappa}_h((\bar{\pi}_h\beta)\nabla)v\|_\Omega\,.\end{aligned}$$

The second term on the right-hand side is simply bounded by

$$\begin{aligned}\|\mu_T^{1/2}\bar{\varkappa}_h((\bar{\pi}_h\beta)\nabla)v\|_\Omega &\leq \|\beta\|_{\infty,\Omega}\|\mu_T^{1/2}\bar{\varkappa}_h\nabla v\|_\Omega\\ &\leq \|\beta\|_{\infty,\Omega}S(v,v)^{1/2}\,.\end{aligned}$$

The first term can be estimated by the approximation property of $\bar{\pi}_h$ and a local inverse inequality:

$$\begin{aligned}\|((\beta-\bar{\pi}_h\beta)\cdot\nabla)v\|_\Omega &\lesssim \sum_{K\in\mathcal{T}_{2h}}\|\bar{\varkappa}\beta\|_{K,\infty}h_K^{-1}\|v\|_K\\ &\leq \|\beta\|_{W^{1,\infty}(\Omega)}\|v\|_\Omega\,.\end{aligned}$$

This gives

$$\begin{aligned}\|\mu_T^{1/2}\bar{\varkappa}_h(\beta\cdot\nabla)v\|_\Omega &\lesssim \|\beta\|_{\infty,\Omega}S(v,v)^{1/2}+\mu_T^{1/2}\|\beta\|_{W^{1,\infty}(\Omega)}\|v\|_\Omega\\ &\leq C_\beta|\!|\!|\{v,0\}|\!|\!|\end{aligned}$$

□

Note that this result is valid immediately (without any assumptions on β) if the form (4.4) is used since for such a choice this term enters directly in the triple norm by definition.

We end this section with the following a priori estimate for the discrete solution of (5.2):

THEOREM 5.7. *If the solution $u=\{v,p\}$ of (5.1) satisfies $u\in[H^{r+1}(\Omega)]^{d+1}$ then we have the a priori estimate*

$$|\!|\!|u-u_h|\!|\!|\lesssim ah^{r+\frac{1}{2}}\left(\|v\|_{r+1,\Omega}+\|p\|_{r+1,\Omega}\right) \tag{5.14}$$

with

$$a=h^{-1/2}(\mu^{1/2}+\mu_T^{1/2}+\nu_T^{1/2})+h^{1/2}(\sigma^{1/2}+\mu_T^{-1/2}+\nu_T^{-1/2})\,. \tag{5.15}$$

Before proceeding with the proof of this Theorem let us briefly comment on its interpretation. An immediate consequence of the inequality (5.14) is that for convection dominated flow $\mu_T\sim h$ and $\nu_T\sim h$ is the optimal choice of the parameters yielding an h-independent constant a and the (optimal) convergence order of $h^{r+1/2}$. The positive powers of μ_T and ν_T in a_h represent the dissipative character of the subgrid model. It follows that too much dissipation will have a negative effect on the precision. The presence of $\mu_T^{-1/2}$ and $\nu_T^{-1/2}$ in (5.15) is due to the stabilizing effect of the subgrid model: the dissipation of the small scale energy into the unresolved scales avoids artificial energy concentrations on the small scales due to the conservation properties of the Galerkin method. As expected, precision deteriorates for small values of μ_T and ν_T due to spurious oscillations.

Proof. In the standard fashion we decompose the error in $u - u_h = \eta + \xi$ in an interpolation part $\eta = u - j_h u$ and a projection part $\xi = j_h u - u_h$. Clearly, $|||\eta||| \leq Cah^{r+1/2}$ using the interpolation Lemma 5.4 and the asymptotic bound for the stabilization term of Lemma 5.5. Consider now the discrete error ξ. By coercivity (Lemma 5.2) and the Galerkin orthogonality property (Lemma 5.3) we have

$$\begin{aligned}|||\xi|||^2 &= A(\xi,\xi) + S_h(\xi,\xi)\\ &= A(\eta,\xi) + S_h(j_h u,\xi)\,.\end{aligned}$$

The second term on the right-hand side is bounded by applying the Cauchy-Schwarz inequality followed by Lemma 5.5

$$\begin{aligned}S_h(j_h u,\xi) &\leq S(j_h u, j_h u)^{1/2} S(\xi,\xi)^{1/2}\\ &\lesssim (\mu^{1/2} + \nu_T^{1/2}) h^r (\|v\|_{r+1,\Omega} + \|p\|_{r+1,\Omega}) |||\xi|||\,.\end{aligned}$$

For the first term on the right-hand side we have using the Cauchy-Schwarz inequality and integration by parts writing ξ^p and ξ^v for the discrete pressure and velocity error, respectively:

$$A(\eta,\xi) \leq |||\eta|||\,|||\xi||| - (p - j_h p, \operatorname{div}\xi^v)_\Omega - (v - j_h v, \nabla\xi^p)_\Omega - (\beta\otimes(v - j_h v), \nabla\xi^v)_\Omega\,.$$

We now use the orthogonality property of the quasi interpolation operator to obtain upper bounds:

$$\begin{aligned}|(p - j_h p, \operatorname{div}\xi^v)_\Omega| &= |(p - j_h p, \operatorname{div}\xi^v - \overline{\pi}\operatorname{div}\xi^v)_\Omega|\\ &\leq \|\mu_T^{-1/2}(p - j_h p)\|_\Omega\,\|\mu_T^{1/2}(\operatorname{div}\xi^v - \overline{\pi}\operatorname{div}\xi^v)\|_\Omega\\ &\leq \|\mu_T^{-1/2}(p - j_h p)\|_\Omega\, S(\xi,\xi)^{1/2}\,,\\ |(v - j_h v, \nabla\xi^p)_\Omega| &= (v - j_h v, \nabla\xi^p - \overline{\pi}\nabla\xi^p)_\Omega\\ &\leq \|\nu_T^{-1/2}(v - j_h v)\|_\Omega\,\|\nu_T^{1/2}(\nabla\xi^p - \overline{\pi}\nabla\xi^p)\|_\Omega\\ &\leq \|\nu_T^{-1/2}(v - j_h v)\|_\Omega\, S(\xi,\xi)^{1/2}\,,\\ |(\beta\otimes(v - j_h v), \nabla\xi^v)_\Omega| &= |(v - j_h v, (\beta\cdot\nabla)\xi^v - \overline{\pi}(\beta\cdot\nabla)\xi^v)_\Omega|\\ &\leq \|\mu_T^{-1/2}(v - j_h v)\|_\Omega\, S(\xi,\xi)^{1/2}\,.\end{aligned}$$

In summary, we get

$$\begin{aligned}A(\eta,\xi) &\leq |||\eta|||\,|||\xi||| + (\|\mu_T^{-1/2}(p - j_h p)\|_\Omega + \|(\nu_T^{-1/2} + \mu_T^{-1/2})(v - j_h v)\|_\Omega) S(\xi,\xi)^{1/2}\\ &\leq \Big(|||\eta||| + \|\mu_T^{-1/2}(p - j_h p)\|_\Omega + \|(\nu_T^{-1/2} + \mu_T^{-1/2})(v - j_h v)\|_\Omega\Big)\, |||\xi|||\,.\end{aligned}$$

The assertion follows using the interpolation properties (5.8) and (5.9) of the quasi interpolant j_h. □ We proceed and prove that the pressure also has optimal convergence properties in the L^2-norm.

LEMMA 5.8. *Let $u = \{v, p\}$ be the solution of (5.1) and $u_h = \{v_h, p_h\}$ the solution of (5.2) then there holds*

$$\|p - p_h\|_\Omega \lesssim a|||u - u_h|||\,,$$

where $a = \sigma^{1/2} + |\beta|\sigma^{-1/2} + \mu^{1/2} + \mu_T^{1/2} + \nu_T^{-1/2} h$.

Proof. Following [10], by the surjectivity of the divergence operator there exists $v_p \in [H_0^1(\Omega)]^d$ such that $p - p_h = \operatorname{div} v_p$ and $\|v_p\|_{1,\Omega} \lesssim \|p - p_h\|_\Omega$. By the H^1-stability property of the quasi interpolant j_h we then have

$$\|j_h v_p\|_{1,\Omega} \lesssim \|p - p_h\|_\Omega \, . \tag{5.16}$$

Consider now the equality $p - p_h = \operatorname{div} v_p$ this gives

$$\|p - p_h\|_\Omega^2 = (p - p_h, \operatorname{div} v_p)_\Omega \, .$$

We now subtract $j_h v_p$ from v_p in the right-hand side and use the Galerkin orthogonality property in Lemma 5.3 for the test function $\{j_h v_p, 0\}$:

$$\begin{aligned}\|p - p_h\|_\Omega^2 = (p - p_h, \operatorname{div}(v_p - j_h v_p))_\Omega - (\mu \nabla (v - v_h), \nabla j_h v_p)_\Omega \\ +(\sigma(v - v_h), j_h v_p)_\Omega + (\beta \otimes (v - v_h), \nabla (j_h v_p))_\Omega - S_h(u - u_h, \{j_h v_p, 0\}) \, .\end{aligned}$$

We estimate the resulting parts separately. For the first term we integrate by parts and use the orthogonality property (5.7) of the quasi interpolation operator j_h to obtain

$$\begin{aligned}(p - p_h, \operatorname{div}(v_p - j_h v_p))_\Omega &= (\nabla (p - p_h), v_p - j_h v_p)_\Omega \\ &= (\bar{\varkappa}_h \nabla (p - p_h), v_p - j_h v_p)_\Omega \\ &\leq S_h(\{0, p - p_h\}, \{0, p - p_h\})^{1/2} \|\nu_T^{-1/2} (v_p - j_h v_p)\|_\Omega \\ &\lesssim \nu_T^{-1/2} h \|u - u_h\| \|v_p\|_{1,\Omega} \\ &\lesssim \nu_T^{-1/2} h \|u - u_h\| \, \|p - p_h\|_\Omega \, ,\end{aligned}$$

where we used the stability property of v_p in the last inequality. Furthermore, we have

$$\begin{aligned}&(\sigma(v - v_h), j_h v_p)_\Omega + (\beta \otimes (v - v_h), \nabla (j_h v_p))_\Omega \\ &= (\sigma(v - v_h), j_h v_p)_\Omega - (v - v_h, (\beta \cdot \nabla) j_h v_p)_\Omega \\ &\leq (\sigma^{1/2} + |\beta| \sigma^{-1/2}) \|u - u_h\| \, \|j_h v_p\|_{1,\Omega}.\end{aligned}$$

Similarly we obtain after application of the Cauchy-Schwarz inequality and (5.5),

$$\begin{aligned}&(\mu \nabla (v - v_h), \nabla j_h v_p)_\Omega - S_h(u - u_h, \{j_h v_p, 0\}) \\ &\leq \|\mu^{1/2} \nabla (v - v_h)\|_\Omega \|\mu^{1/2} \nabla j_h v_p\|_\Omega + S_h(u - u_h, u - u_h)^{1/2} S_h(\{j_h v_p, 0\}, \{j_h v_p, 0\})^{1/2} \\ &\leq \|u - u_h\| \, (\mu^{1/2} + \mu_T^{1/2}) \|j_h v_p\|_{1,\Omega}.\end{aligned}$$

Collecting terms and using (5.16) gives the assertion. ∎

COROLLARY 5.9. *For the solution of (5.2) there holds*

$$\|p_h\|_\Omega^2 \lesssim \|u_h\|^2 = A(u_h, u_h) + S_h(u_h, u_h).$$

Hence the pressure is unique.

Proof. Modifying the proof of Lemma 5.8 we have

$$\|p_h\|_\Omega \lesssim a \|u_h\| \, ,$$

and we conclude applying Lemma 5.2. ∎

5.1. Realistic regularities. The aim of the smoothness assumptions above is to show that the discretization allows for the quasi optimal a priori error estimates that are characteristic for stabilized methods. However, for the case of turbulent flow this may seem overly optimistic and we will therefore discuss what we may prove rigorously in the case where the pressure is only in $H^1(\Omega)$ and the velocities are in $[H^2(\Omega)]^d$. The regularity of the pressure is only necessary for the upper bound of the stabilizing term of Lemma 5.5. The lower regularity will give the modified upper bound

$$S_h(j_h u, j_h u)^{1/2} \lesssim \mu_T^{1/2} h \|v\|_{2,\Omega} + \nu_T^{1/2} \|p\|_{1,\Omega} .$$

Clearly now the estimates will be dominated by the term $\nu_T^{1/2}\|p\|_{1,\Omega}$ which leads to a convergence order of only $h^{1/2}$ for the previous optimal choice of ν_T. It is tempting to decrease the stabilization of the pressure to $\nu_T \sim h^2$ in order to recover optimality of the estimate. However, it follows from the proof of Theorem 5.7 that one then also has to increase the control of the incompressibility condition, demanding

$$\|\bar{\varkappa}_h \mathrm{div}\, v_h\|_\Omega \leq S(u_h, u_h),$$

so that

$$\begin{aligned} |(p - j_h p, \mathrm{div}\, \xi^v)_\Omega| &= |(p - j_h p, \mathrm{div}\, \xi^v - \overline{\pi}\, \mathrm{div}\, \xi^v)_\Omega| \\ &\leq \|p - j_h p\|_\Omega \, S(u_h, u_h). \end{aligned}$$

Numerical experiments indicate that this introduces excessive damping of the small scales so it is questionable whether this strategy is advisable.

In the case of low local Peclet number, i.e. $|\beta| h < \mu$, and if $\{v, p\} \in [H^2(\Omega)]^d \times H^1(\Omega)$, one easily shows that the choice $\mu_T = 0$ and $\nu_T \sim h^2$ leads to optimal a priori error estimates in the energy norm by Theorem 5.7. An error estimate in the L^2-norm for the velocities may then be recovered using a standard Nitsche duality argument.

6. Relation to classical stabilized methods. The VMS approach for large eddy simulations was inspired by the residual-free bubble techniques which in their turn were introduced as a theoretical background for the streamline diffusion method. In this last section, we will try to close the loop by showing the relation between the local projection method (LPS) analyzed in this paper and the Galerkin Least-Squares (GLS) method or the residual-free bubble method. A key feature of the proposed method is the weak consistency: the fact that the stabilization enjoys the right asymptotic behavior without strong consistency allows us to decouple the stabilization of the pressure and the velocities and even more importantly allows us to decouple the stabilization from time-stepping terms and source terms. However, to show the relation to the GLS we will reintroduce the strong consistency. To this end consider the full differential operator in the stabilization

$$\rho(u) := \sigma v + \mathrm{div}\,(\beta \otimes v) - \mu \Delta v + \nabla p\, .$$

This gives the subgrid viscosity

$$S_{gls}(u_h, \varphi) := (\mu_T \bar{\varkappa}_h \rho(u_h), \bar{\varkappa}_h \rho(\varphi))_h + (\mu_T \bar{\varkappa}_h \mathrm{div}\, v_h, \bar{\varkappa}_h \mathrm{div}\, \psi)_\Omega\, , \tag{6.1}$$

where $(\cdot,\cdot)_h := \sum_K (\cdot,\cdot)_K$. To make the formulation strongly consistent we perturb the right hand side and obtain

$$A(u_h, \varphi) + S_{gls}(u_h, \varphi) = (f, \psi + \mu_T \bar{\varkappa}_h \rho(\varphi))_h \quad \forall \varphi \in X_h. \tag{6.2}$$

The consistency follows noting that for the exact solution u we have

$$\begin{aligned} S_{gls}(u,\varphi) - (f, \mu_T \bar{\varkappa}_h \rho(\varphi))_h &= (\mu_T \bar{\varkappa}_h \rho(u), \bar{\varkappa}_h \rho(\varphi))_h - (\bar{\varkappa}_h f, \mu_T \bar{\varkappa}_h \rho(\varphi))_h \\ &= (\mu_T \bar{\varkappa}_h (\rho(U) - f), \bar{\varkappa}_h \rho(\varphi))_h \\ &= 0. \end{aligned}$$

We have thus reformulated the local projection method as a GLS formulation with the stabilization acting only on the small scales. Recalling the form (3.8), this may be interpreted as solving for the fine scales using GLS and then stabilizing the large scales using the fine scale information as in a residual-free bubble approach. An important difference however is that the local projection approach using (6.1) do not impose any artificial boundary conditions on the fine scale solution contrary to the case of residual-free bubbles. Therefore, no spatial restrictions are posed on small scale interaction. This should be a definite advantage for nonlinear problems.

We will outline how to prove an a priori error estimate for (6.1). In fact, after minor modifications, Theorem 5.7 remains true. We first note that

$$\begin{aligned} S_{gls}(u - j_h u, u - j_h u) = &(\mu_T \bar{\varkappa}_h \rho(u - j_h u_h), \bar{\varkappa}_h \rho(u - j_h u))_h \\ &+(\mu_T \bar{\varkappa}_h \mathrm{div}\,(v - j_h v_h), \bar{\varkappa}_h \mathrm{div}\,(v - j_h v))_\Omega \,, \end{aligned}$$

has the right asymptotic, which is immediate assuming optimal approximation for the second derivatives. The key observation that has to be made is then that for the viscosity term in the expression $A(\eta, \xi)$ there holds

$$(\mu \nabla \eta^v, \nabla \xi^v) = (\eta^v, -\mu \Delta \xi^v)_h + \sum_K (\mu \eta^v, \nabla \xi^v \cdot n)_{\partial K} \,.$$

The first part now is added to the pressure term, the convection term and the low order term to form the full residual.

$$\begin{aligned} (\eta^v, \rho(\xi))_h &= (\eta^v, \bar{\varkappa}_h \rho(\xi))_h \\ &\leq \|\mu_T^{-1/2}(v - j_h v)\|_\Omega \, S_{gls}(\xi,\xi)^{1/2} \\ &\leq \|\mu_T^{-1/2}(v - j_h v)\|_\Omega \, |\!|\!|\xi|\!|\!|_{gls} \,, \end{aligned}$$

where $|\!|\!| \cdot |\!|\!|_{gls}$ is the analogous norm to $|\!|\!| \cdot |\!|\!|$ where S_h is replaced by S_{gls}. For the second term we have using a Cauchy-Schwarz inequality followed by a trace inequality and an inverse inequality (recalling that $\mu < h|\beta|$)

$$\begin{aligned} \sum_K (\mu \eta^v, \nabla \xi^v \cdot n)_{\partial K} &\lesssim \sum_K (h^{-1} \|\eta^v\|_K^2 + h \|\nabla \eta^v\|_K^2)^{1/2} \\ &\quad \cdot \; (\mu h^{-1} \|\mu^{1/2} \nabla \xi^v\|_K^2 + \mu h \|\mu^{1/2} \nabla \xi^v\|_{1,K}^2)^{1/2} \\ &\lesssim h^{r+1/2} |\!|\!|\xi|\!|\!|_{gls} \,. \end{aligned}$$

7. Concluding remarks. We have proposed and analyzed a stabilized finite element method based on local projections. We have shown that the method can be formulated in a multiscale setting, hence rigorously establishing a link between stabilized methods and the variational multiscale method (VMS) for Navier-Stokes equations. Moreover, we discussed how the choices of subgrid viscosities may influence the precision of the computation. To assure stability of the large scales a sufficient condition on the characteristic length scale of the subgrid model for a linearized model problem is established for the case of high Reynolds number. This condition coincides with the condition for optimal-order convergence for the stabilized method when the underlying exact solution is smooth. We hope that this contribution will give additional insight in the close relationship between VMS and stabilized finite element methods. Numerical simulations will be reported in a forthcoming paper.

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